

MATHCOUNTS TOOLBOX

Facts, Formulas and Tricks

I. PRIME NUMBERS from 1 through 100 (**1 is not prime!**)

2	3	5	7
11	13	17	19
	23	29	
	31	37	
41	43	47	
	53	59	
	61	67	
71	73	79	
	83	89	
	97		

II. FRACTIONS**DECIMALS****PERCENTS**

$\frac{1}{2}$	.5	50 %
$\frac{1}{3}$	$\bar{.3}$	33. $\bar{3}$ %
$\frac{2}{3}$	$\bar{.6}$	66. $\bar{6}$ %
$\frac{1}{4}$	.25	25 %
$\frac{3}{4}$	.75	75 %
$\frac{1}{5}$	.2	20 %
$\frac{2}{5}$	.4	40 %
$\frac{3}{5}$	.6	60 %
$\frac{4}{5}$	.8	80 %
$\frac{1}{6}$	$\bar{.16}$	16. $\bar{6}$ %
$\frac{5}{6}$	$\bar{.83}$	83. $\bar{3}$ %
$\frac{1}{8}$	.125	12.5 %
$\frac{3}{8}$	.375	37.5 %
$\frac{5}{8}$	.625	62.5 %
$\frac{7}{8}$	.875	87.5 %
$\frac{1}{9}$	$\bar{.1}$	11. $\bar{1}$ %
$\frac{1}{10}$	.1	10 %
$\frac{1}{11}$	$\bar{.09}$	9. $\bar{09}$ %
$\frac{1}{12}$	$\bar{.083}$	8. $\bar{3}$ %
$\frac{1}{16}$	.0625	6.25 %
$\frac{1}{20}$	.05	5 %
$\frac{1}{25}$	.04	4 %
$\frac{1}{50}$	.02	2 %

III. PERFECT SQUARES AND PERFECT CUBES

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$
$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$19^2 = 361$	$20^2 = 400$
$21^2 = 441$	$22^2 = 484$	$23^2 = 529$	$24^2 = 576$	$25^2 = 625$
$1^3 = 1$	$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	$5^3 = 125$
$6^3 = 216$	$7^3 = 343$	$8^3 = 512$	$9^3 = 729$	$10^3 = 1000$

IV. SQUARE ROOTS

$$\begin{array}{lllll} \sqrt{1} = 1 & \sqrt{2} \approx 1.414 & \sqrt{3} \approx 1.732 & \sqrt{4} = 2 & \sqrt{5} \approx 2.236 \\ \sqrt{6} \approx 2.449 & \sqrt{7} \approx 2.646 & \sqrt{8} \approx 2.828 & \sqrt{9} = 3 & \sqrt{10} \approx 3.162 \end{array}$$

V. FORMULAS

Perimeter:

Triangle $p = a + b + c$

Square $p = 4s$

Rectangle $p = 2l + 2w$

Circle (*circumference*) $c = 2\pi r$

$c = \pi d$

Volume:

Cube $V = s^3$

Rectangular Prism $V = lwh$

Cylinder $V = \pi r^2 h$

Cone $V = (\frac{1}{3})\pi r^2 h$

Sphere $V = (\frac{4}{3})\pi r^3$

Pyramid $V = (\frac{1}{3})(\text{area of base})h$

Area:

Rhombus $A = (\frac{1}{2})d_1 d_2$

Square $A = s^2$

Rectangle $A = lw = bh$

Parallelogram $A = bh$

Trapezoid $A = (\frac{1}{2})(b_1 + b_2)h$

Circle $A = \pi r^2$

Triangle $A = (\frac{1}{2})bh$

Right Triangle $A = (\frac{1}{2})l_1 l_2$

Equilateral Triangle $A = (\frac{\sqrt{3}}{4})s^2$

Total Surface Area:

Cube $T = 6s^2$

Rectangular Prism $T = 2lw + 2lh + 2wh$

Cylinder $T = 2\pi r^2 + 2\pi rh$

Sphere $T = 4\pi r^2$

Lateral Surface Area:

Rectangular Prism $L = (2l + 2w)h$

Cylinder $L = 2\pi rh$

Distance = Rate $\times$ Time

Slope of a Line with Endpoints (x_1, y_1) and (x_2, y_2) : $\text{slope} = m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$

Distance Formula: distance between two points or length of segment with endpoints (x_1, y_1) and (x_2, y_2)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula: midpoint of a line segment given two endpoints (x_1, y_1) and (x_2, y_2)

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Circles:

Length of an arc = $\left(\frac{x}{360} \right) (2\pi r)$, where x is the measure of the central angle of the arc

Area of a sector = $\left(\frac{x}{360} \right) (\pi r^2)$, where x is the measure of the central angle of the sector

Combinations (number of groupings when the order of the items in the groups does not matter):

Number of combinations = $\frac{N!}{R!(N-R)!}$, where N = # of total items and R = # of items being chosen

Permutations (number of groupings when the order of the items in the groups matters):

Number of permutations = $\frac{N!}{(N-R)!}$, where N = # of total items and R = # of items being chosen

Length of a Diagonal of a Square = $s\sqrt{2}$

Length of a Diagonal of a Cube = $s\sqrt{3}$

Length of a Diagonal of a Rectangular Solid = $\sqrt{x^2 + y^2 + z^2}$, with dimensions x , y and z

Number of Diagonals for a Convex Polygon with N Sides = $\frac{N(N-3)}{2}$

Sum of the Measures of the Interior Angles of a Regular Polygon with N Sides = $(N-2)180$

Heron's Formula:

For **any triangle** with side lengths a , b and c , $Area = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

Pythagorean Theorem: (Can be used with **all right triangles**)

$a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse

Pythagorean Triples: Integer-length sides for right triangles form Pythagorean Triples – the largest number must be on the hypotenuse. Memorizing the bold triples will also lead to other triples that are multiples of the original.

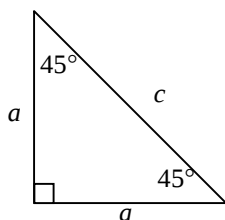
3	4	5		5	12	13		7	24	25
6	8	10		10	24	26		8	15	17
9	12	15		15	36	39		9	40	41

Special Right Triangles:

$45^\circ - 45^\circ - 90^\circ$

hypotenuse = $\sqrt{2}$ (leg) = $a\sqrt{2}$

leg = $\frac{\text{hypotenuse}}{\sqrt{2}} = \frac{c}{\sqrt{2}}$

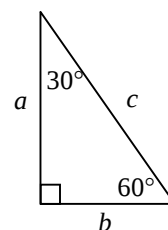


$30^\circ - 60^\circ - 90^\circ$

hypotenuse = 2(shorter leg) = $2b$

longer leg = $\sqrt{3}$ (shorter leg) = $b\sqrt{3}$

shorter leg = $\frac{\text{longer leg}}{\sqrt{3}} = \frac{\text{hypotenuse}}{2}$



Geometric Mean: $\frac{a}{x} = \frac{x}{b}$ therefore, $x^2 = ab$ and $x = \sqrt{ab}$

Regular Polygon: Measure of a central angle = $\frac{360}{n}$, where n = number of sides of the polygon

Measure of vertex angle = $180 - \frac{360}{n}$, where n = number of sides of the polygon

Ratio of Two Similar Figures: If the ratio of the measures of corresponding side lengths is $A:B$, then the ratio of the perimeters is $A:B$, the ratio of the areas is $A^2 : B^2$ and the ratio of the volumes is $A^3 : B^3$.

Difference of Two Squares: $a^2 - b^2 = (a - b)(a + b)$

Example: $12^2 - 9^2 = (12 - 9)(12 + 9) = 3 \cdot 21 = 63$
 $144 - 81 = 63$

Determining the Greatest Common Factor (GCF): 5 Methods

1. Prime Factorization (Factor Tree) – Collect all common factors

2. Listing all Factors

3. Multiply the two numbers and divide by the Least Common Multiple (LCM)

Example: to find the GCF of 15 and 20, multiply $15 \times 20 = 300$,
then divide by the LCM, 60. The GCF is 5.

4. Divide the smaller number into the larger number. If there is a remainder, divide the remainder into the divisor until there is no remainder left. The last divisor used is the GCF.

Example: $180 \overline{)385}$ $25 \overline{)180}$ $5 \overline{)25}$
 $\underline{360}$ $\underline{175}$ $\underline{25}$
25 5 0 5 is the GCF of 180 and 385

5. Single Method for finding both the GCF and LCM

Put both numbers in a lattice. On the left, put ANY divisor of the two numbers and put the quotients below the original numbers. Repeat until the quotients have no common factors except 1 (relatively prime). Draw a “boot” around the left-most column and the bottom row. Multiply the vertical divisors to get the GCF. Multiply the “boot” numbers (vertical divisors and last-row quotients) to get the LCM.

	40	140
2	20	70

	40	140
2	20	70
10		

	40	140
2	20	70
10	2	7

The GCF is $2 \times 10 = 20$

The LCM is

$2 \times 10 \times 2 \times 7 = 280$

VI. DEFINITIONS

Real Numbers: all rational and irrational numbers

Rational Numbers: numbers that can be written as a ratio of two integers

Irrational Numbers: non-repeating, non-terminating decimals; can't be written as a ratio of two integers
(i.e. $\sqrt{7}$, π)

Integers: {..., -3, -2, -1, 0, 1, 2, 3, ...}

Whole Numbers: {0, 1, 2, 3, ...}

Natural Numbers: {1, 2, 3, 4, ...}

Common Fraction: a fraction in lowest terms (Refer to “Forms of Answers” in the *MATHCOUNTS School Handbook* for a complete definition.)

Equation of a Line:

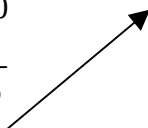
Standard form: $Ax + By = C$ with slope = $-\frac{A}{B}$

Slope-intercept form: $y = mx + b$ with slope = m and y -intercept = b

Regular Polygon: a convex polygon with all equal sides and all equal angles

Negative Exponents: $x^{-n} = \frac{1}{x^n}$ and $\frac{1}{x^{-n}} = x^n$

Systems of Equations:

$x + y = 10$		$8 + y = 10$	(8, 2) is the solution of the system
$x - y = 6$		$y = 2$	
$2x = 16$			
$x = 8$			

Mean = Arithmetic Mean = Average

Mode = the number(s) occurring the most often; there may be more than one

Median = the middle number when written from least to greatest

If there is an even number of terms, the median is the average of the two middle terms.

Range = the difference between the greatest and least values

Measurements:

1 mile = 5280 feet

1 square foot = 144 square inches

1 square yard = 9 square feet

1 cubic yard = 27 cubic feet

VII. PATTERNS

Divisibility Rules:

Number is divisible by 2: last digit is 0,2,4,6 or 8

3: sum of digits is divisible by 3

4: two-digit number formed by the last two digits is divisible by 4

5: last digit is 0 or 5

6: number is divisible by **both** 2 and 3

8: three-digit number formed by the last 3 digits is divisible by 8

9: sum of digits is divisible by 9

10: last digit is 0

Sum of the First N Odd Natural Numbers = N^2

Sum of the First N Even Natural Numbers = $N^2 + N = N(N + 1)$

Sum of an Arithmetic Sequence of Integers: $\frac{N}{2} \times (\text{first term} + \text{last term})$, where N = amount of numbers/terms in the sequence

Find the digit in the units place of a particular power of a particular integer

Find the pattern of units digits: 7^1 ends in 7

7^2 ends in 9

(pattern repeats 7^3 ends in 3

every 4 exponents) 7^4 ends in 1

7^5 ends in 7

Divide 4 into the given exponent and compare the remainder with the first four exponents.
(a remainder of 0 matches with the exponent of 4)

Example: What is the units digit of 7^{22} ?

$22 \div 4 = 5 \text{ r. } 2$, so the units digit of 7^{22} is the same as the units digit of 7^2 , which is 9.

VIII. FACTORIALS (“ $n!$ ” is read “ n factorial”)

$n! = (n) \times (n-1) \times (n-2) \times \dots \times (2) \times (1)$ Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$0! = 1$

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = 720$

$7! = 5040$

Notice $\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1}} = 30$

IX. PASCAL’S TRIANGLE

Pascal’s Triangle Used for Probability:

Remember that the first row is row zero (0). Row 4 is 1 4 6 4 1. This can be used to determine the different outcomes when flipping four coins.

1	4	6	4	1
way to get	ways to get	ways to get	ways to get	way to get
4 heads 0 tails	3 heads 1 tail	2 heads 2 tails	1 head 3 tails	0 heads 4 tails

For the Expansion of $(a + b)^n$, use numbers in Pascal’s Triangle as coefficients.

1	$(a + b)^0 = 1$
1 1	$(a + b)^1 = a + b$
1 2 1	$(a + b)^2 = a^2 + 2ab + b^2$
1 3 3 1	$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
1 4 6 4 1	$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
1 5 10 10 5 1	$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

For 2^n , add all the numbers in the n^{th} row. (Remember the triangle starts with row 0.)

1	$2^0 = 1$
1 1	$2^1 = 1 + 1 = 2$
1 2 1	$2^2 = 1 + 2 + 1 = 4$
1 3 3 1	$2^3 = 1 + 3 + 3 + 1 = 8$
1 4 6 4 1	$2^4 = 1 + 4 + 6 + 4 + 1 = 16$
1 5 10 10 5 1	$2^5 = 1 + 5 + 10 + 10 + 5 + 1 = 32$

X. SQUARING A NUMBER WITH A UNITS DIGIT OF 5

$(n5)^2 = \underline{n \times (n+1)} \underline{2} \underline{5}$, where n represents the block of digits before the units digit of 5

Examples:

$$\begin{aligned}(35)^2 &= \underline{3 \times (3+1)} \underline{2} \underline{5} \\ &= \underline{3 \times (4)} \underline{2} \underline{5} \\ &= \underline{12} \underline{2} \underline{5} \\ &= 1,225\end{aligned}$$

$$\begin{aligned}(125)^2 &= \underline{12 \times (12+1)} \underline{2} \underline{5} \\ &= \underline{12 \times (13)} \underline{2} \underline{5} \\ &= \underline{156} \underline{2} \underline{5} \\ &= 15,625\end{aligned}$$

XI. BASES

Base 10 = decimal – only uses digits 0 – 9

Base 2 = binary – only uses digits 0 – 1

Base 8 = octal – only uses digits 0 – 7

Base 16 = hexadecimal – only uses digits 0 – 9, A – F (where A=10, B=11, ..., F=15)

Changing from Base 10 to Another Base:

What is the base 2 representation of 125 (or “125 base 10” or “125₁₀”)?

We know $125 = 1(10^2) + 2(10^1) + 5(10^0) = 100 + 20 + 5$, but what is it equal to in base 2?

$$125_{10} = ?(2^n) + ?(2^{n-1}) + \dots + ?(2^0)$$

The largest power of 2 in 125 is $64 = 2^6$, so we now know our base 2 number will be:

$$?(2^6) + ?(2^5) + ?(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0) \text{ and it will have 7 digits of 1's and/or 0's.}$$

Since there is **one** 64, we have: $1(2^6) + ?(2^5) + ?(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$

We now have $125 - 64 = 61$ left over, which is **one** $32 = 2^5$ and 29 left over, so we have:

$$1(2^6) + 1(2^5) + ?(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$$

In the left-over 29, there is **one** $16 = 2^4$, with 13 left over, so we have:

$$1(2^6) + 1(2^5) + 1(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$$

In the left-over 13, there is **one** $8 = 2^3$, with 5 left over, so we have:

$$1(2^6) + 1(2^5) + 1(2^4) + 1(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$$

In the left-over 5, there is **one** $4 = 2^2$, with 1 left over, so we have:

$$1(2^6) + 1(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + ?(2^1) + ?(2^0)$$

In the left-over 1, there is **no** $2 = 2^1$, so we still have 1 left over, and our expression is:

$$1(2^6) + 1(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + ?(2^0)$$

The left-over 1 is **one** 2^0 , so we finally have:

$$1(2^6) + 1(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) = 1111101_2$$

Now try **What is the base 3 representation of 105?**

The largest power of 3 in 105 is $81 = 3^4$, so we now know our base 3 number will be: $?(3^4) + ?(3^3) + ?(3^2) + ?(3^1) + ?(3^0)$ and will have 5 digits of 2's, 1's, and/or 0's.

Since there is **one** 81, we have: $1(3^4) + ?(3^3) + ?(3^2) + ?(3^1) + ?(3^0)$

In the left-over $105 - 81 = 24$, there is **no** $27 = 3^3$, so we still have 24 and the expression: $1(3^4) + 0(3^3) + ?(3^2) + ?(3^1) + ?(3^0)$

In the left-over 24, there are **two** 9's (or 3^2 's), with 6 left over, so we have:

$1(3^4) + 0(3^3) + 2(3^2) + ?(3^1) + ?(3^0)$

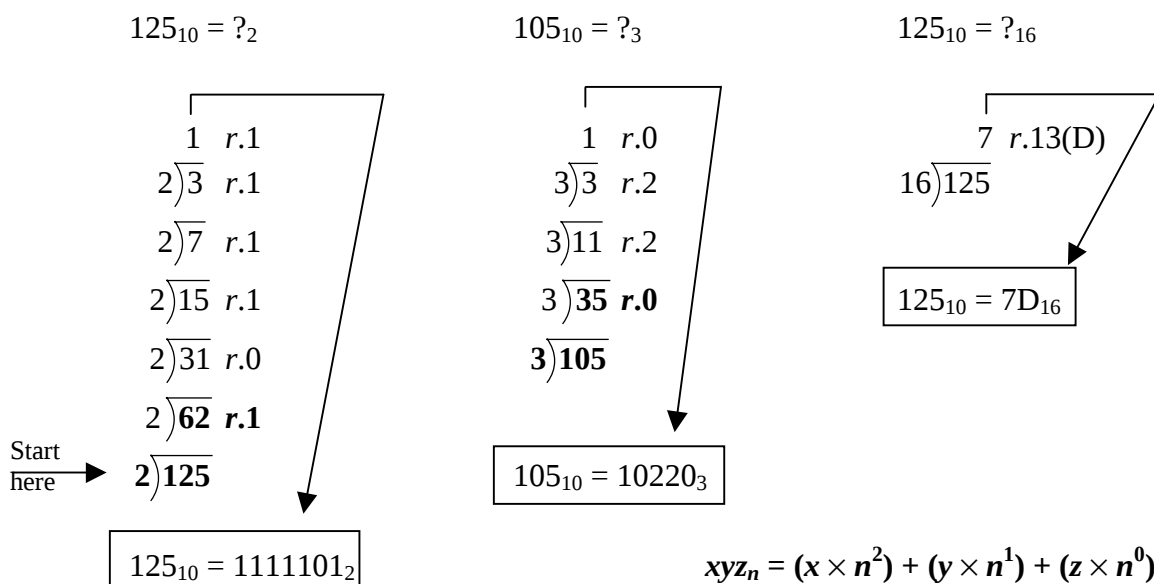
In the left-over 6, there are **two** 3's (or 3^1 's), with 0 left over, so we have:

$1(3^4) + 0(3^3) + 2(3^2) + 2(3^1) + ?(3^0)$

Since there is nothing left over, we have **no** 1's (or 3^0 's), so our final expression is:

$1(3^4) + 0(3^3) + 2(3^2) + 2(3^1) + 0(3^0) = 10220_3$

The following is another fun algorithm for converting base 10 numbers to other bases:



Notice: Everything in bold shows the first division operation. The first remainder will be the last digit in the base n representation, and the quotient is then divided again by the desired base. The process is repeated until a quotient is reached that is less than the desired base. At that time, the final quotient and remainders are read downward.

XII. FACTORS

Determining the Number of Factors of a Number: First find the prime factorization (include the 1 if a factor is to the first power). *Increase* each exponent by 1 and multiply these new numbers together.

Example: How many factors does 300 have?

The prime factorization of 300 is $2^2 \times 3^1 \times 5^2$. Increase each of the exponents by 1 and multiply these new values: $(2+1) \times (1+1) \times (2+1) = 3 \times 2 \times 3 = 18$. So 300 has 18 factors.

Finding the Sum of the Factors of a Number:

Example: What is the sum of the factors of 10,500?

(From the prime factorization $2^2 \times 3^1 \times 5^3 \times 7^1$, we know 10,500 has $3 \times 2 \times 4 \times 2 = 48$ factors.)

The sum of these 48 factors can be calculated from the prime factorization, too:

$(2^0 + 2^1 + 2^2)(3^0 + 3^1)(5^0 + 5^1 + 5^2 + 5^3)(7^0 + 7^1) = 7 \times 4 \times 156 \times 8 = 34,944$.